PROBLEMS ON CONTINUTY

Question 1:(5 MARK)

Prove that the function $f(x)=5x-3$ is continuous at $x=0$, at $x=-3$ and at $x=5$.

Answer

The given function is $f(x)=5x-3$

At $x=0$, $f(0)=5\times 0-3=3$

$\lim_{x\to 0} f(x)=\lim_{x\to 0}(5x-3)=5\times 0-3=-3$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, *f* is continuous at *x* = 0

At $x=-3$, $f(-3)=5\times(-3)-3=-18$

$\lim_{x\to -3} f(x)=\lim_{x\to -3}(5x-3)=5\times(-3)-3=-18$

$\therefore \lim_{x\to -3} f(x)=f(-3)$

Therefore, *f* is continuous at *x* = −3

At $x=5$, $f(x)=f(5)=5\times 5-3=25-3=22$

$\lim_{x\to 5} f(x)=\lim_{x\to 5}(5x-3)=5\times 5-3=22$

$\therefore \lim_{x\to 5} f(x)=f(5)$

Therefore, *f* is continuous at *x* = 5

Question 2: **(5 MARK)**

Examine the continuity of the function $f(x)=2x^2-1$ at $x=3$.

Answer

The given function is $f(x)=2x^2-1$

At $x=3$, $f(x)=f(3)=2\times 3^2-1=17$

$\lim_{x\to 3} f(x)=\lim_{x\to 3}(2x^2-1)=2\times 3^2-1=17$

$\therefore \lim_{x\to 3} f(x)=f(3)$

Thus, *f* is continuous at *x* = 3

Question 3: **(10 MARK)**

Examine the following functions for continuity.

(a) $f(x)=x-5$ (b) $f(x)=\frac{1}{x-5}, x\neq 5$

(c) $f(x)=\frac{x^2-25}{x+5}, x\neq -5$ (d) $f(x)=|x-5|$

Answer

(a) The given function is $f(x)=x-5$

It is evident that *f* is defined at every real number *k* and its value at *k* is *k* − 5.

It is also observed that, $\lim_{x\to k} f(x)=\lim_{x\to k}(x-5)=k-5=f(k)$

$\therefore \lim_{x\to k} f(x)=f(k)$

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

(b) The given function is $f(x)=\frac{1}{x-5}, x\neq 5$

For any real number *k* ≠ 5, we obtain

$\lim_{x\to k} f(x)=\lim_{x\to k}\frac{1}{x-5}=\frac{1}{k-5}$

$\text{Also, } f(k)=\frac{1}{k-5} \qquad (\text{As } k\neq 5)$

$\therefore \lim_{x\to k} f(x)=f(k)$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(c) The given function is $f(x)=\frac{x^2-25}{x+5}, x\neq -5$

For any real number *c* ≠ −5, we obtain

$$\lim_{x \to c} f(x) = \lim_{x \to c} \frac{x^2 - 25}{x+5} = \lim_{x \to c} \frac{(x+5)(x-5)}{x+5} = \lim_{x \to c}(x-5) = (c-5)$$

$$\text{Also, } f(c) = \frac{(c+5)(c-5)}{c+5} = (c-5) \quad (\text{as } c \neq -5)$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Hence, *f* is continuous at every point in the domain of *f* and therefore, it is a continuous function.

(d) The given function is $f(x) = |x-5| = \begin{cases} 5-x, \text{ if } x < 5 \\ x-5, \text{ if } x \geq 5 \end{cases}$

This function *f* is defined at all points of the real line.

Let *c* be a point on a real line. Then, *c* < 5 or *c* = 5 or *c* > 5

Case I: *c* < 5

Then, *f* (*c*) = 5 − *c*

$$\lim_{x \to c} f(x) = \lim_{x \to c}(5-x) = 5-c$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers less than 5.

Case II : *c* = 5

Then, $f(c) = f(5) = (5-5) = 0$

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5}(5-x) = (5-5) = 0$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5}(x-5) = 0$$

$$\therefore \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

Therefore, *f* is continuous at *x* = 5

Case III: *c* > 5

$$\text{Then, } f(c) = f(5) = c-5$$

$$\lim_{x \to c} f(x) = \lim_{x \to c}(x-5) = c-5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all real numbers greater than 5.

Hence, *f* is continuous at every real number and therefore, it is a continuous function.

Question 4: **(5 MARK)**

Prove that the function $f(x)=x^n$ is continuous at $x = n$, where n is a positive integer.

Answer

The given function is $f(x) = x^n$

It is evident that f is defined at all positive integers, n, and its value at n is n^n.

$$\text{Then, } \lim_{x\to n} f(n)=\lim_{x\to n}\left(x^n\right)=n^n$$

$$\therefore \lim_{x\to n} f(x)=f(n)$$

Therefore, f is continuous at n, where n is a positive integer.

Question 5: **(10 MARK)**

Is the function f defined by

$$f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$$

continuous at $x = 0$? At $x = 1$? At $x = 2$?

Answer

The given function f is $f(x)=\begin{cases} x, & \text{if } x\le 1 \\ 5, & \text{if } x>1 \end{cases}$

At $x = 0$,

It is evident that f is defined at 0 and its value at 0 is 0.

$$\text{Then, } \lim_{x\to 0} f(x)= \lim_{x\to 0} x=0$$

$$\therefore \lim_{x\to 0} f(x)=f(0)$$

Therefore, f is continuous at $x = 0$

At $x = 1$,

f is defined at 1 and its value at 1 is 1.

The left hand limit of f at $x = 1$ is,

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} x=1$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(5) = 5$$

$$\therefore \lim_{x\to1^-} f(x) \neq \lim_{x\to1^+} f(x)$$

Therefore, *f* is not continuous at *x* = 1

At *x* = 2,

f is defined at 2 and its value at 2 is 5.

Then, $\lim_{x\to2} f(x) = \lim_{x\to2}(5) = 5$

$$\therefore \lim_{x\to2} f(x) = f(2)$$

Therefore, *f* is continuous at *x* = 2

Question 5: **(10 MARK)**

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

It is evident that the given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line. Then, three cases arise.

(i) *c* < 2

(ii) *c* > 2

(iii) *c* = 2

Case (i) *c* < 2

Then, $f(c) = 2c + 3$

$$\lim_{x\to c} f(x) = \lim_{x\to c}(2x+3) = 2c+3$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* < 2

Case (ii) *c* > 2

Then, $f(c) = 2c - 3$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (2x - 3) = 2c - 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that *x* > 2

Case (iii) *c* = 2

Then, the left hand limit of *f* at *x* = 2 is,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7$$

The right hand limit of *f* at *x* = 2 is,

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x - 3) = 2 \times 2 - 3 = 1$$

It is observed that the left and right hand limit of *f* at *x* = 2 do not coincide.

Therefore, *f* is not continuous at *x* = 2

Hence, *x* = 2 is the only point of discontinuity of *f*.

Question 6: **(10 MARK)**

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} |x| + 3, \text{if } x \leq -3 \\ -2x, \text{if } -3 < x < 3 \\ 6x + 2, \text{if } x \geq 3 \end{cases}$$

Answer

$$f(x) = \begin{cases} |x| + 3 = -x + 3, \text{ if } x \leq -3 \\ -2x, \text{ if } -3 < x < 3 \\ 6x + 2, \text{ if } x \geq 3 \end{cases}$$

The given function *f* is

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c < -3$, then $f(c) = -c + 3$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (-x + 3) = -c + 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x < -3$

Case II:

If $c = -3$, then $f(-3) = -(-3) + 3 = 6$

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} (-x + 3) = -(-3) + 3 = 6$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} (-2x) = -2 \times (-3) = 6$$

$$\therefore \lim_{x \to -3} f(x) = f(-3)$$

Therefore, f is continuous at $x = -3$

Case III:

If $-3 < c < 3$, then $f(c) = -2c$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (-2x) = -2c$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous in $(-3, 3)$.

Case IV:

If $c = 3$, then the left hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (-2x) = -2 \times 3 = -6$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (6x + 2) = 6 \times 3 + 2 = 20$$

It is observed that the left and right hand limit of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $c > 3$, then $f(c) = 6c + 2$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (6x + 2) = 6c + 2$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 3$

Hence, $x = 3$ is the only point of discontinuity of f.

Question 7: **(10 MARK)**

Find all points of discontinuity of f, where f is defined by

$$f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x=0\end{cases}$$

Answer

The given function *f* is $$f(x)=\begin{cases}\frac{|x|}{x} & \text{if } x \neq 0 \\ 0, & \text{if } x=0\end{cases}$$

It is known that, $x<0 \Rightarrow |x|=-x$ and $x>0 \Rightarrow |x|=x$

Therefore, the given function can be rewritten as

$$f(x)=\begin{cases}\frac{|x|}{x}=\frac{-x}{x}=-1 & \text{if } x<0 \\ 0, & \text{if } x=0 \\ \frac{|x|}{x}=\frac{x}{x}=1, & \text{if } x>0\end{cases}$$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c<0$, then $f(c)=-1$

$$\lim_{x \to c} f(x)=\lim_{x \to c}(-1)=-1$$

$$\therefore \lim_{x \to c} f(x)=f(c)$$

Therefore, *f* is continuous at all points $x<0$

Case II:

If $c=0$, then the left hand limit of *f* at $x=0$ is,

$$\lim_{x \to 0^-} f(x)=\lim_{x \to 0^-}(-1)=-1$$

The right hand limit of *f* at $x=0$ is,

$$\lim_{x \to 0^+} f(x)=\lim_{x \to 0^+}(1)=1$$

It is observed that the left and right hand limit of *f* at $x=0$ do not coincide.

Therefore, *f* is not continuous at $x=0$

Case III:

$\text{If } c > 0, \text{ then } f(c) = 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (1) = 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* > 0

Hence, *x* = 0 is the only point of discontinuity of *f*.

Question 8: **(10 MARK)**

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} x+1, & \text{if } x \geq 1 \\ x^2+1, & \text{if } x < 1 \end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$\text{If } c < 1, \text{ then } f(c) = c^2 + 1 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 1

Case II:

$\text{If } c = 1, \text{ then } f(c) = f(1) = 1 + 1 = 2$

The left hand limit of *f* at *x* = 1 is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 1^2 + 1 = 2$

The right hand limit of *f* at *x* = 1 is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2$

$\therefore \lim_{x \to 1} f(x) = f(1)$

Therefore, *f* is continuous at *x* = 1

Case III:

$\text{If } c > 1, \text{ then } f(c) = c + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x + 1) = c + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x > 1$

Hence, the given function *f* has no point of discontinuity.

Question 9: **(10 MARK)**

Find all points of discontinuity of *f*, where *f* is defined by

$$f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$$

Answer

The given function *f* is $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

$\text{If } c < 2, \text{ then } f(c) = c^3 - 3 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c} (x^3 - 3) = c^3 - 3$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x < 2$

Case II:

$\text{If } c = 2, \text{ then } f(c) = f(2) = 2^3 - 3 = 5$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3) = 2^3 - 3 = 5$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1) = 2^2 + 1 = 5$

$\therefore \lim_{x \to 2} f(x) = f(2)$

Therefore, *f* is continuous at $x = 2$

Case III:

If $c > 2$, then $f(c) = c^2 + 1$

$\lim_{x \to c} f(x) = \lim_{x \to c} (x^2 + 1) = c^2 + 1$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x > 2$

Thus, the given function *f* is continuous at every point on the real line.

Hence, *f* has no point of discontinuity.

Question 10: **(10 MARK)**

Is the function defined by

$$f(x) = \begin{cases} x + 5, & \text{if } x \le 1 \\ x - 5, & \text{if } x > 1 \end{cases}$$

a continuous function?

Answer

The given function is $f(x) = \begin{cases} x + 5, & \text{if } x \le 1 \\ x - 5, & \text{if } x > 1 \end{cases}$

The given function *f* is defined at all the points of the real line.

Let *c* be a point on the real line.

Case I:

If $c < 1$, then $f(c) = c + 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (x + 5) = c + 5$

$\therefore \lim_{x \to c} f(x) = f(c)$

Therefore, *f* is continuous at all points *x*, such that $x < 1$

Case II:

If $c = 1$, then $f(1) = 1 + 5 = 6$

The left hand limit of *f* at $x = 1$ is,

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x + 5) = 1 + 5 = 6$

The right hand limit of *f* at $x = 1$ is,

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 5) = 1 - 5 = -4$

It is observed that the left and right hand limit of *f* at $x = 1$ do not coincide.

Therefore, *f* is not continuous at $x = 1$

Case III:

$$\text{If } c > 1, \text{ then } f(c) = c - 5 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(x - 5) = c - 5$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points *x*, such that $x > 1$

Thus, from the above observation, it can be concluded that $x = 1$ is the only point of discontinuity of *f*.

Question 11: **(10 MARK)**

Discuss the continuity of the function *f*, where *f* is defined by

$$f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$$

Answer

The given function is $f(x) = \begin{cases} 3, & \text{if } 0 \le x \le 1 \\ 4, & \text{if } 1 < x < 3 \\ 5, & \text{if } 3 \le x \le 10 \end{cases}$

The given function is defined at all points of the interval [0, 10].

Let *c* be a point in the interval [0, 10].

Case I:

$$\text{If } 0 \le c < 1, \text{ then } f(c) = 3 \text{ and } \lim_{x \to c} f(x) = \lim_{x \to c}(3) = 3$$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, *f* is continuous in the interval [0, 1).

Case II:

$$\text{If } c = 1, \text{ then } f(3) = 3$$

The left hand limit of *f* at $x = 1$ is,

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-}(3) = 3$$

The right hand limit of f at $x = 1$ is,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4) = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case III:

If $1 < c < 3$, then $f(c) = 4$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (4) = 4$

$$\therefore \lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (1, 3).

Case IV:

If $c = 3$, then $f(c) = 5$

The left hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (4) = 4$$

The right hand limit of f at $x = 3$ is,

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (5) = 5$$

It is observed that the left and right hand limits of f at $x = 3$ do not coincide.

Therefore, f is not continuous at $x = 3$

Case V:

If $3 < c \le 10$, then $f(c) = 5$ and $\lim_{x \to c} f(x) = \lim_{x \to c} (5) = 5$

$$\lim_{x \to c} f(x) = f(c)$$

Therefore, f is continuous at all points of the interval (3, 10].

Hence, f is not continuous at $x = 1$ and $x = 3$

Question 12: **(10 MARK)**

Discuss the continuity of the function f, where f is defined by

$$f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$$

Answer

The given function is $f(x)=\begin{cases}2x, & \text{if } x<0\\ 0, & \text{if } 0\le x\le 1\\ 4x, & \text{if } x>1\end{cases}$

The given function is defined at all points of the real line.

Let *c* be a point on the real line.

Case I:

$\text{If } c<0, \text{ then } f(c)=2c$

$\lim_{x\to c} f(x)=\lim_{x\to c}(2x)=2c$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points *x*, such that *x* < 0

Case II:

$\text{If } c=0, \text{ then } f(c)=f(0)=0$

The left hand limit of *f* at *x* = 0 is,

$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(2x)=2\times 0=0$

The right hand limit of *f* at *x* = 0 is,

$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(0)=0$

$\therefore \lim_{x\to 0} f(x)=f(0)$

Therefore, *f* is continuous at *x* = 0

Case III:

$\text{If } 0<c<1, \text{ then } f(x)=0 \text{ and } \lim_{x\to c} f(x)=\lim_{x\to c}(0)=0$

$\therefore \lim_{x\to c} f(x)=f(c)$

Therefore, *f* is continuous at all points of the interval (0, 1).

Case IV:

$\text{If } c=1, \text{ then } f(c)=f(1)=0$

The left hand limit of *f* at *x* = 1 is,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(0)=0$

The right hand limit of f at $x = 1$ is,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(4x) = 4\times1 = 4$$

It is observed that the left and right hand limits of f at $x = 1$ do not coincide.

Therefore, f is not continuous at $x = 1$

Case V:

$$\text{If } c < 1, \text{ then } f(c) = 4c \text{ and } \lim_{x\to c} f(x) = \lim_{x\to c}(4x) = 4c$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points x, such that $x > 1$

Hence, f is not continuous only at $x = 1$

Question 13: **(10 MARK)**

Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$.

Answer

The given function f is $f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$

If f is continuous at $x = 3$, then

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x) = f(3) \qquad \ldots(1)$$

Also,

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-}(ax+1) = 3a+1$$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+}(bx+3) = 3b+3$$

$$f(3) = 3a+1$$

Therefore, from (1), we obtain

$3a+1=3b+3=3a+1$

$\Rightarrow 3a+1=3b+3$

$\Rightarrow 3a=3b+2$

$\Rightarrow a=b+\frac{2}{3}$

Therefore, the required relationship is given by, $a=b+\frac{2}{3}$

Question 14: **(10 MARK)**

For what value of λ is the function defined by

$$f(x)=\begin{cases}\lambda\left(x^2-2x\right), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$$

continuous at x = 0? What about continuity at x = 1?

Answer

The given function f is $f(x)=\begin{cases}\lambda\left(x^2-2x\right), & \text{if } x\le 0\\ 4x+1, & \text{if } x>0\end{cases}$

If f is continuous at x = 0, then

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)$

$\Rightarrow \lim\limits_{x\to 0^-} \lambda\left(x^2-2x\right)=\lim\limits_{x\to 0^+}(4x+1)=\lambda\left(0^2-2\times 0\right)$

$\Rightarrow \lambda\left(0^2-2\times 0\right)=4\times 0+1=0$

$\Rightarrow 0=1=0,$ which is not possible

Therefore, there is no value of λ for which f is continuous at x = 0

At $x = 1$,

$f(1) = 4x + 1 = 4 \times 1 + 1 = 5$

$$\lim_{x \to 1}(4x+1) = 4\times1+1 = 5$$

$$\therefore \lim_{x \to 1} f(x) = f(1)$$

Therefore, for any values of λ, f is continuous at $x = 1$

Question 15: **(5 MARK)**

Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral point.

Here $[x]$ denotes the greatest integer less than or equal to x.

Answer

The given function is $g(x) = x - [x]$

It is evident that g is defined at all integral points.

Let n be an integer.

Then,

$$g(n) = n - [n] = n - n = 0$$

The left hand limit of f at $x = n$ is,

$$\lim_{x \to n^-} g(x) = \lim_{x \to n^-}(x - [x]) = \lim_{x \to n^-}(x) - \lim_{x \to n^-}[x] = n - (n-1) = 1$$

The right hand limit of f at $x = n$ is,

$$\lim_{x \to n^+} g(x) = \lim_{x \to n^+}(x - [x]) = \lim_{x \to n^+}(x) - \lim_{x \to n^+}[x] = n - n = 0$$

It is observed that the left and right hand limits of f at $x = n$ do not coincide.

Therefore, f is not continuous at $x = n$

Hence, g is discontinuous at all integral points.

Question 16: **(5 MARK)**

Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at x = p?

Answer

The given function is $f(x) = x^2 - \sin x + 5$

It is evident that *f* is defined at *x* = p

At $x = \pi$, $f(x) = f(\pi) = \pi^2 - \sin\pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$

Consider $\lim_{x\to\pi} f(x) = \lim_{x\to\pi}\left(x^2 - \sin x + 5\right)$

Put $x = \pi + h$

If $x \to \pi$, *then it is evident that* $h \to 0$

$$\begin{aligned}
\therefore \lim_{x\to\pi} f(x) &= \lim_{x\to\pi}\left(x^2 - \sin x + 5\right) \\
&= \lim_{h\to0}\left[(\pi+h)^2 - \sin(\pi+h) + 5\right] \\
&= \lim_{h\to0}(\pi+h)^2 - \lim_{h\to0}\sin(\pi+h) + \lim_{h\to0}5 \\
&= (\pi+0)^2 - \lim_{h\to0}\left[\sin\pi\cos h + \cos\pi\sin h\right] + 5 \\
&= \pi^2 - \lim_{h\to0}\sin\pi\cos h - \lim_{h\to0}\cos\pi\sin h + 5 \\
&= \pi^2 - \sin\pi\cos0 - \cos\pi\sin0 + 5 \\
&= \pi^2 - 0\times1 - (-1)\times0 + 5 \\
&= \pi^2 + 5
\end{aligned}$$

$\therefore \lim_{x\to\pi} f(x) = f(\pi)$

Therefore, the given function *f* is continuous at *x* = π

Question 17: **(10 MARK)**

$\therefore \lim_{x\to0^-} f(x) = f(0) = \lim_{x\to0^+} f(x)$

Therefore, *f* is continuous at *x* = 0

Determine if *f* defined by

$$f(x) = \begin{cases} x^2\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is a continuous function?

Answer

$$f(x) = \begin{cases} x^2\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

The given function *f* is

It is evident that *f* is defined at all points of the real line.

Let *c* be a real number.

Case I:

If $c \neq 0$, then $f(c) = c^2 \sin\frac{1}{c}$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^2 \sin\frac{1}{x}\right) = \left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c}\sin\frac{1}{x}\right) = c^2 \sin\frac{1}{c}$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, *f* is continuous at all points $x \neq 0$

Case II:

If $c = 0$, then $f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

It is known that, $-1 \leq \sin\frac{1}{x} \leq 1,\ x \neq 0$

$$\Rightarrow -x^2 \leq \sin\frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \leq \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \leq \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \leq 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

Similarly, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, *f* is continuous at $x = 0$

From the above observations, it can be concluded that *f* is continuous at every point of the real line.

Thus, *f* is a continuous function

Question 18: **(5 MARK)**

Find the values of *k* so that the function *f* is continuous at the indicated point.

$$f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases} \qquad \text{at } x=5$$

Answer

The given function *f* is $f(x)=\begin{cases} kx+1, & \text{if } x\le 5 \\ 3x-5, & \text{if } x>5 \end{cases}$

The given function *f* is continuous at *x* = 5, if *f* is defined at *x* = 5 and if the value of *f* at *x* = 5 equals the limit of *f* at *x* = 5

It is evident that *f* is defined at *x* = 5 and $f(5)=kx+1=5k+1$

$$\lim_{x\to 5^-} f(x)=\lim_{x\to 5^+} f(x)=f(5)$$

$$\Rightarrow \lim_{x\to 5^-}(kx+1)=\lim_{x\to 5^+}(3x-5)=5k+1$$

$$\Rightarrow 5k+1=15-5=5k+1$$

$$\Rightarrow 5k+1=10$$

$$\Rightarrow 5k=9$$

$$\Rightarrow k=\frac{9}{5}$$

Therefore, the required value of k is $\frac{9}{5}$.

Question 19: **(10 MARK)**

Find the values of *a* and *b* such that the function defined by

$$f(x)=\begin{cases} 5, & \text{if } x\le 2 \\ ax+b, & \text{if } 2<x<10 \\ 21, & \text{if } x\ge 10 \end{cases}$$

is a continuous function.

Answer

The given function f is
$$f(x)=\begin{cases}5, & \text{if } x\le 2\\ ax+b, & \text{if } 2<x<10\\ 21, & \text{if } x\ge 10\end{cases}$$

It is evident that the given function f is defined at all points of the real line.

If f is a continuous function, then f is continuous at all real numbers.

In particular, f is continuous at $x = 2$ and $x = 10$

Since f is continuous at $x = 2$, we obtain

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$$

$$\Rightarrow \lim_{x\to 2^-}(5)=\lim_{x\to 2^+}(ax+b)=5$$

$$\Rightarrow 5=2a+b=5$$

$$\Rightarrow 2a+b=5 \qquad \ldots(1)$$

Since f is continuous at $x = 10$, we obtain

$$\lim_{x\to 10^-} f(x)=\lim_{x\to 10^+} f(x)=f(10)$$

$$\Rightarrow \lim_{x\to 10^-}(ax+b)=\lim_{x\to 10^+}(21)=21$$

$$\Rightarrow 10a+b=21=21$$

$$\Rightarrow 10a+b=21 \qquad \ldots(2)$$

On subtracting equation (1) from equation (2), we obtain

$8a = 16$

$\Rightarrow a = 2$

By putting $a = 2$ in equation (1), we obtain

$2 \times 2 + b = 5$

$\Rightarrow 4 + b =$
$\Rightarrow b = 1$

Therefore, the values of a and b for which f is a continuous function are 2 and 1 respectively.